JULY 1902

ANNALS OF MATHEMATICS

(Founded by Ormond Stone)

PUBLISHED UNDER THE AUSPICES OF HARVARD UNIVERSITY

Second Series Vol. 3 No. 4

FOR SALE BY
THE PUBLICATION OFFICE OF HARVARD UNIVERSITY
2 University Hall, Cambridge, Mass., U. S. A.

London: Longmans, Green & Co.
39 Paternoster Row

Leipzig: Otto Harrassowitz
Querstrasse 14

NOTE ON A TWISTED CURVE CONNECTED WITH AN INVOLUTION OF PAIRS OF POINTS IN A PLANE.*

By H. S. White.

1. A Method for Generating some Twisted Curves. The points of a plane are said to be in involution when they are so related in sets of n points that any one point of a set determines all the other $n-1$ points of that set. The simplest involutions, and the best known, are those where each set contains only 2 points. Of involutions of higher order so much is known as that they are all rational; that is, that between the points of a plane and the sets of points of any involution a one-to-one relation can be established.† But for orders higher than 2 the classification and reduction to types have not yet been given. Hence we shall understand for the present by "involution" always an involution of order 2, one whose points are grouped in pairs.

Bertini showed that involutions of second order are of four distinct types, every such involution falling into some one of these four classes.‡ Two of the same type are usually not projectively equivalent, but each can be transformed into the other by some Cremona transformation. As the representative of each type he chooses the simplest. The first type is that in which collineation or projection effects an exchange of the points of each pair. Such a collineation evidently must be a perspectivity, or central projection. The second type includes as many distinct species as there are possible orders of curves, one for every integer above 3. Among these we consider that of lowest order, in which paired points lie on rays passing through a fixed point, and are conjugate with respect to a fixed conic.§ These two alone are to be used in the present note.

The notion of an involution in a plane may be used to generate twisted curves, just as the notion of involution on a line has been used (and may be

* Read before the Chicago Section of the American Mathematical Society at its meeting January 3, 1902.

† G. Castelnuovo, *Mathematische Annalen*, vol. 44 (1894), p. 125.

‡ Ricerche sulle trasformazioni univoche involutorie nel piano. *Annali di matematica*, ser. 2, vol. 8 (1877), p. 254.

§ Compare the system of intersections of the circles of a net, used as an illustration, by Professor Bôcher in the *Annals of Mathematics*, ser. 2, vol. 3 (1902), pp. 49-52, and especially the first footnote on p. 52.

used further) to generate plane curves. To make this explicit, recall the method proposed by Steiner, and elaborated by Schroeter, for generating a cubic curve in the plane. A quadric involution on a straight line, it will be remembered, consists of pairs of points harmonic with respect to two fixed points, either real or imaginary, on the line. Two pairs of points determine fully such an involution, since there is only a single pair of points separating harmonically two given pairs. Any third pair of points, in order to belong to the same involution, must satisfy a single condition. Hence if three pairs of points are taken at random in a plane, and these are projected from a variable center in the plane upon any line, the requirement that their projections shall form 3 pairs in quadric involution will subject the variable center to a single condition; *i. e.* the locus of that point is a definite curve. In this case the locus is a general cubic curve.

In the same way, if a certain number of pairs of points in a plane suffice to determine an involution of a specified type, then one additional pair of points, in order to belong in the same involution, must satisfy two conditions. Assume now, in space of 3 dimensions, pairs of points one more than sufficient: project them upon a plane from a variable center, and require the projections to form pairs in an involution of the specified type. This will be equivalent to 2 conditions, restricting the variable center to motion along some definite twisted curve.

In this way involutions of order 2 can give rise to an infinite variety of algebraic twisted curves, and may be found to lead to important properties of those curves. Indeed two species of involution in the plane which are equivalent to each other by virtue of some Cremona transformation will usually be connected with twisted curves which are not transformable into one another by any Cremona transformation of three dimensional space.

To exhibit the method I have worked out the following two simplest examples.

2. First Example: the Perspective Involution. In a perspective relation of the points in a plane, the axis o and the center O may be any line and any point. To find the conjugate to any point P_1 (see Fig. 1), produce a line OP_1 to cut the axis o in a point P', and determine a fourth point P_2 on OP_1 harmonic to O, P_1, and P'. All the pairs of points P_1P_2 constitute then an involution of the first type. Otherwise, two such pairs may be taken arbitrarily, and from them the center, the axis, and consequently all other pairs can be found. Suppose A_1 and A_2, B_1 and B_2 to denote given

pairs, then any third pair $P_1 P_2$ must have their joining line pass through the point O where the line $A_1 A_2$ meets $B_1 B_2$. This is the first condition. Again, this center O determines two points A' and B' where the axis o must intersect $A_1 A_2$ and $B_1 B_2$; the second condition is that the center O and axis o shall separate harmonically the points P_1 and P_2.

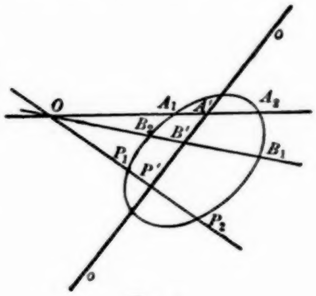

FIG. 1.

Assume in space three pairs of points $A_1 A_2$, $B_1 B_2$, $C_1 C_2$ (no four in any plane). If the projections of the 3 lines a, b, c which join these several pairs are to meet in one point of a plane, the center of projection X must lie on a line intersecting all three: a, b, and c. That is, X must lie on a ruled quadric having a, b, c as directrices. This first condition restricts X to a quadric surface.

The second condition is satisfied if the six projecting rays XA_1, XA_2, XB_1, . . . XC_2, lie on a quadric cone. For then the generator which passes through X, and its polar plane with respect to the cone, will project from X into the required center and axis of perspectivity, inasmuch as their sections by any plane will separate harmonically the projections of each pair of given points, *e. g.* of A_1 and A_2. Conversely we see that this condition is not only sufficient, but also necessary for the point X. We have to find therefore the locus of the vertex of a cone passing through 6 given points. That this is a surface of the fourth order* can be shown as follows.

A quadric surface is fixed by 9 points. Hence through the 6 given points there pass all the quadrics of a linear system with 3 arbitrary parameters:

$$\phi_0 + \lambda_1 \phi_1 + \lambda_2 \phi_2 + \lambda_3 \phi_3 = 0.$$

If X is to be a conical point upon one of these quadrics, 4 conditions must be satisfied, *viz.* the vanishing of the 4 partial derivatives:†

$$\begin{array}{c} \phi_{01}(x) + \lambda_1 \phi_{11}(x) + \lambda_2 \phi_{21}(x) + \lambda_3 \phi_{31}(x) = 0, \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \phi_{04}(x) + \lambda_1 \phi_{14}(x) + \lambda_2 \phi_{24}(x) + \lambda_3 \phi_{34}(x) = 0. \end{array}$$

* The "Surface of Weddle." See full discussion in Cayley's *Collected Works*, vol. 7, p. 160 sqq.; and a paper by Chasles, *Comptes Rendus*, vol. 52 (1861) pp. 1157–1162.

† The quadrics ϕ_0, ϕ_1, etc., are supposed to be homogeneous in the coordinates x_1, x_2, x_3, x_4. Partial derivatives are then denoted by added subscripts. Thus the abbreviation $\phi_{21}(x)$ means $\frac{\partial}{\partial x_1} \phi_2 (x_1 x_2, x_3, x_4)$.

Eliminating from these 4 equations, linear in the (x), the parameters $\lambda_1, \lambda_2, \lambda_3$, we find the equation of a quartic locus for X. Notice that it is the Jacobian of all quadric surfaces that pass through the 6 fixed points.

The intersection of a quadric and a quartic is in general an octavic curve. But this quartic contains entire lines, among them the 15 which join two and two the 6 fixed points: and 3 of these are the directrices a, b, c of the quadric. Hence the proper curve is a quintic, and cuts every directrix in 4 points, but every generator of the quadric (*i. e.* those of the system opposite to a, b, c) in only 1 point. This quintic belongs obviously to the class a_0' of Noether,* a sub-species not particularly described by Halphen,† but included under his second species of quintics. All directrices of the quadric surface are four-fold secants of the curve, so that its projection from any point of the curve upon a plane is a plane quartic with a triple point, proving that the quintic is of deficiency zero, *i. e.* it is rational. Generators of the second system meet it in only one point each, hence it can have no actual double point. Finally, a counting of constants makes it probable that every quintic of Noether's class a_0' can be generated by the above method: if this is true, a geometric proof would disclose an interesting property of the curve.

3. Second Example: Involution of Harmonic Conjugates on Rays Cutting a Conic. Five pairs of conjugate points with respect to a conic completely determine that conic. Hence 5 pairs of points suffice to determine an involution of the second type. But these 5 pairs are not perfectly arbitrary; they are subject to the restriction that the 5 lines joining the respective pairs must have a common point, which we may call a *radiant point*. Any sixth pair of points, in order to belong in the same involution as the first five, must satisfy 2 conditions: first, their joining line must pass through the radiant point; and second, the two points must be conjugate with respect to the same conic which separates harmonically the first 5 pairs. In Fig. 2 O is the radiant point, $A_1 A_2$, $B_1 B_2$, etc., the 5 pairs that determine the involution, o the conic which separates each pair harmonically, and $P_1 P_2$ any sixth pair in the involution.

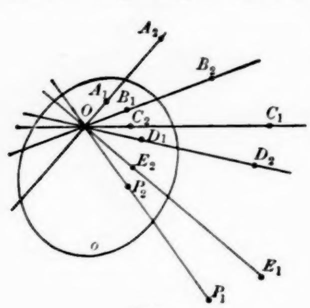

FIG. 2.

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* Zur Grundlegung der Theorie d. algebraischen Raumcurven. *Abhndlgn. der kgl. Akad. d. Wissenschaften zu Berlin*, 1882, Anhang, p. 83.

† Sur la classification des courbes gauches algébriques. *Jour. de l'Ecole polytechnique*, vol. 52 (1882), p. 162.

To secure a variable center of projection we have therefore to assume in space, not 6 pairs of points taken at random, but one pair on each of 6 directrices of a ruled quadric surface. Then a center of projection X anywhere upon that quadric will give a set of projections satisfying the first condition, for the 6 directrices will project into lines meeting upon the generator that passes through X. As to the second condition, some quadric surface separating harmonically each of the 6 given pairs of points must be a cone having X for its vertex.* Each pair of points, as being conjugates, supplies one linear condition for the coefficients of the equation of that quadric, leaving 3 free parameters $(10 - 6 - 1 = 3)$. Here therefore the reasoning of §2 applies again, and the locus of the vertex X is a quartic surface, the Jacobian of 4 quadrics that have generally no common point. This Jacobian must evidently contain each one of the 12 given points, but need not contain any of their join lines. The curve of intersection is accordingly of order 8 and is of the species a_9 of Noether (*l. c.*, p. 96), or Halphen's 1st species (*l. c.*, p. 165).

By counting constants we may judge that every curve of this species lies on at least one Jacobian surface, and contains 24 pairs of points separated harmonically by all cones of the system. But that these pairs, or any six of them, will lie on as many of the quarti-secants of the curve is apparently improbable. Hence the curve found from six arbitrary pairs of points on six generators of a quadric by the aid of the concept, involution of the second type, must be of a very special kind under its species. To confirm this, it would be needful to study minutely the system of lines joining corresponding points on the general Jacobian surface.

NORTHWESTERN UNIVERSITY,
EVANSTON, ILLINOIS, FEBRUARY, 1902.

* See Cayley and Chasles, as cited above.

ON SOME CURVES CONNECTED WITH A SYSTEM OF SIMILAR CONICS.

By R. E. Allardice.

1. Introduction. The problem has been proposed by Steiner* of finding the envelope of a system of similar conics circumscribed about a given triangle, and of finding the loci of the centres and foci of the conics of the system. He states that the envelope is a curve of the fourth order having three double points, and gives some of its properties. The problem has been treated by P. H. Schoute in a paper entitled *Application de la transformation par droites symétriques à un problème de Steiner*.† In this paper the author discusses the problem of the envelope in detail by a geometrical method, to which reference will be made hereafter, and gives the order of the locus of centres, of the locus of foci, and of the locus of vertices, and the class of the envelope of asymptotes, of the envelope of axes, and of the envelope of directrices. He states, however, that the envelope of the asymptotes is a curve of the sixth class, but does not point out that it breaks up into two curves, each of the third class. I propose to determine analytically the equations of certain of these curves.

2. Definition of Similar Conics by the Use of Asymptotes. It may be taken as a definition or proved as a theorem, according to the point of view, that two conics are similar when the angles between their asymptotes are equal. When rectangular coordinates are used, two conics with their centres at the origin are similar and similarly situated if the terms of the second degree be the same, or proportional, in the two equations. Now these terms of the second degree represent the (real or imaginary) asymptotes, and the angle between these asymptotes is not altered if the conic be displaced in any way. It may easily be shown that the tangent of the angle between the asymptotes of an ellipse is equal to $2abi/(a^2 + b^2)$, where a and b are the semi-axes, and for a real ellipse this quantity is a pure imaginary whose modulus is less than

* Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander (problem 39 of the supplement), *Gesammelte Werke*, vol. 1, p. 446; Vermischte Sätze und Aufgaben, *ibid.*, vol. 2, p. 675.

† *Bulletin des sciences mathématiques et astronomiques*, ser. 2, vol. 7 (1883), pp. 314–324.

unity. It is obvious that two conics for which this quantity is the same are similar, as the quantity may be expressed in terms of b/a, which in turn can be expressed in terms of the eccentricity.

3. Laguerre's Projective Definition of Angle applied to the Asymptotes. It has been shown by Laguerre and Cayley* that the angle between two straight lines can be expressed in terms of the cross-ratio of the range of four points consisting of the points where the straight lines meet the straight line at infinity and the circular points at infinity. More definitely,† if α be this cross-ratio and θ the angle between the lines, then $\theta = \frac{1}{2}i\log \alpha$; and from this it may easily be shown that $\tan^2\theta = -(1-\alpha)^2/(1+\alpha)^2$. Now the infinitely distant points of the two asymptotes are the points where the conic itself meets the line at infinity. We thus have to consider a variable conic that circumscribes a given triangle and meets the straight line at infinity in two points that form with the circular points at infinity a range of constant cross-ratio. But instead of the line at infinity and the two circular points upon it we may take any straight line with two fixed points on it; and these fixed points will be most conveniently assigned as the points of intersection with the given straight line of a fixed conic circumscribing the given triangle. The special case of the system of similar conics will then be obtained by taking the straight line at infinity and the circumscribed circle for the fixed straight line and fixed conic.

4. The Quadric System of Similar Circumscribed Conics. By means of the theory of invariants we may write down the condition that a straight line meet two conics in pairs of points having a given cross-ratio α.‡

Let

$$u_x \equiv u_1x_1 + u_2x_2 + u_3x_3 = 0$$

be the given straight line;

$$p_x^2 \equiv p_1x_2x_3 + p_2x_3x_1 + p_3x_1x_2 = 0$$

be the fixed circumscribed conic; and

$$\lambda_x^2 \equiv \lambda_1x_2x_3 + \lambda_2x_3x_1 + \lambda_3x_1x_2 = 0$$

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* Laguerre, Note sur la théorie des foyers, *Nouvelles annales de mathématiques*, vol. 12 (1853), p. 64. Cayley, A sixth memoir upon quantics, *Philosophical Transactions*, vol. 149 (1859), pp. 61-90, and *Collected Mathematical Papers*, vol. 2, pp. 561-592. See also Loria, *Teorie geometriche*, where, however, the reference is given to Cayley's fifth, instead of to his sixth memoir.

† Compare Klein, *Nicht-Euklidische Geometrie*, vol. 1, pp. 47-60.

‡ See Clebsch-Lindemann, *Vorlesungen über Geometrie*, vol. 1, p. 281.

be the variable circumscribed conic. The required condition is

$$D'^2(a-1)^2 - DD''(a+1)^2 = 0, \qquad (1)$$

where

$$-\tfrac{1}{2}D = \lambda_1^2u_1^2 + \lambda_2^2u_2^2 + \lambda_3^2u_3^2 - 2\lambda_1\lambda_2u_1u_2 - 2\lambda_2\lambda_3u_2u_3 - 2\lambda_3\lambda_1u_3u_1;$$

$$-\tfrac{1}{2}D'' = p_1^2u_1^2 + p_2^2u_2^2 + p_3^2u_3^2 - 2p_1p_2u_1u_2 - 2p_2p_3u_2u_3 - 2p_3p_1u_3u_1;$$

$$-\tfrac{1}{2}D' = \lambda_1p_1u_1^2 + \lambda_2p_2u_2^2 + \lambda_3p_3u_3^2 - (\lambda_3p_2 + \lambda_2p_3)u_2u_3 - \cdot\cdot\cdot\cdot\cdot .$$

Since according to (1) the parameters λ_1, λ_2, λ_3, which occur linearly in the equation of the conic, are connected by a quadric relation, the conics are said to form a *quadric system*.

5. Locus of Centres of Conics in the Quadric System. Corresponding to the problem of finding the locus of the centres of a system of similar conics we have to find the locus of the pole of u_x with respect to the conic λ_x^2.

If (y_1, y_2, y_3) be the required pole, we have

$$u_1 = \lambda_2y_3 + \lambda_3y_2, \text{ etc.},$$

whence, omitting again a factor of proportionality, we have:

$$\lambda_1 = y_1(u_2y_2 + u_3y_3 - u_1y_1); \quad \text{etc.} \qquad (2)$$

Substituting these values of λ_1, λ_2, λ_3 in (1) we get a curve of the fourth order as the locus of the pole of u_x.

If $(\lambda_1, \lambda_2, \lambda_3)$ be regarded as a parametric point, the locus of this point is a system of conics (for varying values of a) of which the equation is given by (1). This system of conics has double contact with the conic $D = 0$ (an inscribed conic), the chord of contact being $D' = 0$. The relations (2) show that this system of conics is transformed into the system of quartic curves by means of a Cremona quadratic transformation. As such a transformation turns a non-specialized conic into a trinodal quartic, we shall expect to find a system of trinodal quartics, all having double contact with the quartic which replaces the conic $D = 0$.

If we substitute the values of λ_1, λ_2, λ_3 in D, D', D'', we find, putting $P_1 = u_1(p_2u_2 + p_3u_3 - p_1u_1)$, etc.,

$$D = -(u_1y_1 + u_2y_2 + u_3y_3)(u_2y_2 + u_3y_3 - u_1y_1)(u_3y_3 + u_1y_1 - u_2y_2)$$
$$(u_1y_1 + u_2y_2 - u_3y_3),$$

$$D' = P_1u_1y_1^2 + P_2u_2y_2^2 + P_3u_3y_3^2 - (P_1u_2 + P_2u_1)y_1y_2 - \text{etc.},$$

$$D'' = -(P_1p_1 + P_2p_2 + P_3p_3).$$

Hence the equation of the system of quartics is

$$(a-1)^2\,[P_1 u_1 y_1^2 + \ldots - (P_1 u_2 + P_2 u_1)\, y_1 y_2 - \ldots\ldots]^2$$
$$-(a+1)^2(P_1 p_1 + P_2 p_2 + P_3 p_3)(u_1 y_1 + u_2 y_2 + u_3 y_3)(u_2 y_2 + u_3 y_3 - u_1 y_1)$$
$$\ldots\ldots = 0.$$

The fact that $D = 0$ degenerates into four lines suggests a simplification of the equation. Choosing the three components which form a symmetrical set, we transform to a new triangle of reference by means of the equations

$$u_2 y_2 + u_3 y_3 - u_1 y_1 = u_1 x_1, \text{ etc.}$$

The result is:

$$(a-1)^2 u_1 u_2 u_3 (p_1 x_2 x_3 + p_2 x_3 x_1 + p_3 x_1 x_2)^2$$
$$-(a+1)^2 k x_1 x_2 x_3 (u_1 x_1 + u_2 x_2 + u_3 x_3) = 0,$$

where $$k = P_1 p_1 + P_2 p_2 + P_3 p_3.$$

This equation represents a trinodal quartic having its double points at the vertices of the new triangle of reference and having u_x for a double tangent. By proper selection of the five constants involved in a, u_1, u_2, u_3, p_1, p_2, p_3, the equation may be made to represent any trinodal quartic referred to the triangle having its vertices at the double points. In other words, any quartic with nodes at three distinct finite points and touching the line at infinity in the two circular points is a locus of centres of similar conics through three fixed points, for evidently such a locus can be constructed to have seventeen points in common with the given quartic.

6. The Locus of Centres for Particular Values of the Asymptotic Angle. We return to the case of the system of similar conics by putting $p_1 = u_1 = a$, etc.; and the equation becomes

$$(x_2 x_3 \sin A + x_3 x_1 \sin B + x_1 x_2 \sin C)^2$$
$$+ \cot^2\theta \sin A \sin B \sin C\, x_1 x_2 x_3 (x_1 \sin A + x_2 \sin B + x_3 \sin C) = 0,$$

where θ is the angle contained by the asymptotes, a, b and c the sides, A, B and C the angles of the new triangle.

For $\theta = \pi/2$, the case of the equilateral hyperbola, the quartic degenerates into the circumscribed circle, that is, the nine-point circle of the original triangle, taken twice over. (In the more general case, where the constants p_1, u_1, etc., are arbitrary, we get the nine-point conic; and this is otherwise obvious, as in this case $a = -1$, and the conics therefore intersect the straight

line u_x in pairs of points in involution: and hence pass through a fourth fixed point.)

For $\theta = 0$, the quartic degenerates into the straight line at infinity and the sides of the triangle of reference. The straight line at infinity corresponds to proper parabolas and the sides of the triangle of reference to degenerate parabolas consisting of pairs of parallel straight lines.

For $\cot^2\theta = -1$, the case of circumscribed conics through either one of the circular points at infinity, the quartic degenerates into two imaginary conics intersecting in the vertices of the triangle and in the circum-centre.

It may easily be shown that the tangents at the vertex A are real when $\cot^2\theta$ is positive, and when $\cot^2\theta$ is negative and $|\cot^2\theta \sin^2 A| > 4$. When $4 + \cot^2\theta \sin^2 A = 0$, the tangents coincide and A is a cusp: this value of θ gives the three-cusped hypocycloid when the triangle of reference is equilateral.

7. Similar Conics under the Isogonal Transformation. The transformation by *droites symétriques* employed by P. H. Schoute in the paper referred to above is sometimes called the "isogonal transformation" and consists in substituting $(1/x_1, 1/x_2, 1/x_3)$ for (x_1, x_2, x_3). Schoute proves that to a system of similar circumscribed conics corresponds, in this transformation, the system of tangents to a circle concentric with the circumscribed circle. This theorem is very easily proved by means of the projective definition of an angle. To the straight line at infinity corresponds the circumscribed circle, and *vice versa*: hence the circular points at infinity are transformed into each other by the isogonal transformation. It is easily seen that if the points P, Q, R, S be transformed into the points P', Q', R', S', and if A be any vertex of the triangle of reference, then the cross-ratio of the pencil $A.PQRS$ is equal to the cross-ratio of the pencil $A.P'Q'R'S'$. Now let P_∞ and Q_∞ be the points in which one of a system of similar circumscribed conics meets the straight line at infinity, let P and Q be the corresponding points, namely, the points in which the straight line into which the conic is transformed meets the circumcircle: and let I and J be the circular points at infinity. Then the cross-ratio $(A.IJP_\infty Q_\infty)$ is equal to the cross-ratio $(A.JIPQ)$: thus the latter cross-ratio is constant, and thus the angle PAQ is constant. In other words the chord PQ subtends a constant angle at A and therefore touches a fixed circle concentric with the circumscribed circle.

8. Envelope of Asymptotes. The asymptotes of a conic are the tangents from the pole of the straight line at infinity. Instead of this special line we may in the first instance take any line whatever. It will be conven-

ient to use tangential coordinates; and the equation of the variable circumscribed conic becomes:

$$\lambda_1^2 v_1^2 + \lambda_2^2 v_2^2 + \lambda_3^2 v_3^2 - 2\lambda_1\lambda_2 v_1 v_2 - 2\lambda_2\lambda_3 v_2 v_3 - 2\lambda_3\lambda_1 v_3 v_1 = 0,$$

or
$$\lambda_1 v_1 Q_1 + \lambda_2 v_2 Q_2 + \lambda_3 v_3 Q_3 = 0, \tag{1}$$
where
$$Q_1 = \lambda_2 v_2 + \lambda_3 v_3 - \lambda_1 v_1, \text{ etc.}$$

The equation of the pole of the straight line (u_1, u_2, u_3) is

$$\lambda_1 u_1 Q_1 + \lambda_2 u_2 Q_2 + \lambda_3 u_3 Q_3 = 0. \tag{2}$$

Hence we have to find the envelope of the lines common to (1) and (2), subject to the condition (see (1), §4):

$$D'^2(a-1)^2 - DD''(a+1)^2 = 0. \tag{3}$$

Equations (1) and (2) are identically satisfied if we set $\lambda_1, \lambda_2, \lambda_3$ proportional to the cubic functions of the coordinates v:

$$v_1(u_2 v_3 - u_3 v_2)^2, \quad v_2(u_3 v_1 - u_1 v_3)^2, \quad v_3(u_1 v_2 - u_2 v_1)^2.$$

The equation of the envelope, (3), *is then separable into two cubic factors.* These are the following, if we denote as usual $(u_1 v_2 - u_2 v_1)$ by $(uv)_3$, etc.:

$$\begin{aligned} &p_1 u_1 (uv)_2 (uv)_3 (u_2 v_3 + u_3 v_2) + p_2 u_2 (uv)_3 (uv)_1 (u_3 v_1 + u_1 v_3) \\ &+ p_3 u_3 (uv)_1 (uv)_2 (u_1 v_2 + u_2 v_1) \pm h\sqrt{D''}(uv)_1 (uv)_2 (uv_3) = 0, \end{aligned} \tag{4}$$

where h denotes $\dfrac{a+1}{a-1}$. Since interchange of the two asymptotes to any one conic changes the cross-ratio a into $\dfrac{1}{a}$, or $+h$ into $-h$, it is evident that only one of the two asymptotes touches one of these cubic envelopes, the other asymptote touching the other.

The fact that the envelope of asymptotes resolves into two separate curves might have been anticipated from inspection of equation (3): when $a = 1$ the envelope is given by $D = 0$, but for $a = 1$ this is evidently the product of the equations, each doubly counted, of the infinite points on the sides of the original triangle (or on the sides of the second triangle of reference, since pairs of sides meet at infinity). Hence *D must be itself a square of a rational cubic function of v_1, v_2, v_3*, as seen in (4).

The envelope just now considered,

$$\sqrt{D} = 0,$$

has the line at infinity for a triple tangent. For $a = -1$ the line at infinity is a double tangent, with contacts in I and J.* Accordingly for every value of a or h, each of the envelopes of asymptotes must have the line at infinity for a double tangent and the contacts, if determinate, must be in I and J, two distinct points. Since each envelope is of class 3 and has a double tangent with distinct points of contact, it is of order 4 (like the locus of centres) and has three cusps. And since each side of the original triangle is an asymptote for each value of a, each envelope of asymptotes touches all three sides of the triangle. This fact is obvious also from the equation. From counting of constants we may conclude, what can be proven rigorously by geometric considerations:

Every tricuspidal quartic, touching the three sides of a triangle ABC and touching the line at infinity in the two circular points, is an envelope of asymptotes of similar conics circumscribed to the triangle ABC. All such quartics are three-cusped hypocycloids,† and this fact invites further investigation as to their arrangement in pairs when the triangle ABC is given.

9. The Degeneration of the Sextic Envelope of Asymptotes. The reason why each asymptote gives a separate envelope, in other words, why the envelope consists of two curves of the third class, and not of a single curve of the sixth class, is that the coordinates of either of the asymptotes may be expressed rationally in terms of those of the other; for, as I have shown in a recent number of the ANNALS,‡ the asymptotes of a circumscribed conic are isotomically conjugate with respect to the triangle of reference. Thus the tangential equation of the envelope of the asymptote $v_1 x_1 + v_2 x_2 + v_3 x_3 = 0$ may be obtained by writing down the condition that this line and the line $a^2 x_1/v_1 + b^2 x_2/v_2 + c^2 x_3/v_3 = 0$, which is the second asymptote of a circumscribed conic having the former line as one asymptote, should intersect at a given angle; and this condition is of the third degree in v_1, v_2, v_3. The envelope of the second asymptote is obtained from that of the first by changing v_1, v_2, v_3 to $a^2/v_1, b^2/v_2, c^2/v_3$.

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* Compare Huntington and Whittemore: Some curious properties of conics touching the line infinity at one of the circular points. *Bulletin Am. Math. Soc.*, vol. 8 (1901), p. 122.

† *Cf.* E. Duporcq, Sur l'hypocycloïde à trois rebroussements, *Nouvelles annales de mathematiques*, ser. 4, vol. 1 (1901), p. 168.

‡ Ser. 2, vol. 2 (1901), p. 148.

NOTE ON MULTIPLY PERFECT NUMBERS.*

By Jacob Westlund.

In the Annals of Mathematics, ser. 2, vol. 2 (1900/01), p. 103, Dr. D. N. Lehmer proves that no multiply perfect numbers of multiplicity 3, containing less than three distinct primes, exist. In an earlier note on Multiply Perfect Numbers† all numbers of multiplicity 3 of the form $p_1^{\alpha_1}p_2^{\alpha_2}p_3$ were determined. The object of the present note is to determine all numbers of multiplicity 3 of the form $m = p_1^{\alpha_1}p_2p_3p_4$ where p_1, p_2, p_3, p_4 are distinct primes and $p_1 < p_2 < p_3 < p_4$.

Defining a multiply perfect number as one which is an exact divisor of the sum of all its divisors, the quotient being the multiplicity, we have‡ in the present case

$$3 = \frac{p_1 - 1/p_1^{\alpha_1}}{p_1 - 1} \prod_{i=2}^{4} \frac{p_i - 1/p_i}{p_i - 1} = \frac{p_1^{\alpha_1+1} - 1}{p_1^{\alpha_1}(p_1 - 1)} \prod_{i=2}^{4} \frac{p_i + 1}{p_i} \tag{1}$$

and

$$3 < \prod_{i=1}^{4} \frac{p_i}{p_i - 1}. \tag{2}$$

From the inequality (2) we infer that $p_1 = 2$ is the only possible value of p_1, since the maximum value of $\prod_{i=1}^{4} \frac{p_i}{p_i - 1}$ will exceed 3 only for $p_1 = 2$. Hence

$$\frac{3}{2} < \prod_{i=2}^{4} \frac{p_i}{p_i - 1},$$

which gives $p_2 < 7$, *i. e.* the only possible values of p_2 are 3 and 5.

I. Suppose $p_2 = 3$. Then we should have from (1)

$$3 = \frac{2^{\alpha_1+1} - 1}{2^{\alpha_1}} \cdot \frac{4}{3} \cdot \frac{p_3 + 1}{p_3} \cdot \frac{p_4 + 1}{p_4}$$

or

$$9 = \frac{2^{\alpha_1+1} - 1}{2^{\alpha_1-2}} \cdot \frac{p_3 + 1}{p_3} \cdot \frac{p_4 + 1}{p_4}. \tag{3}$$

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* Read before the Chicago Section of the American Mathematical Society, January 3, 1902.

† Westlund, Annals of Mathematics, ser. 2, vol. 2 (1900/01), p. 172.

‡ *Cf.* Lehmer, *l. c.*

From this we infer that

$$\left(\frac{p_3+1}{p_3}\right)^2 > \frac{9\cdot 2^{a_1-2}}{2^{a_1+1}-1} > \frac{9}{8}$$

or

$$\frac{p_3+1}{p_3} > \frac{3}{4}\sqrt{2}\,.$$

Hence $p_3 < 17$, *i. e.* the only possible values of p_3 are 5, 7, 11, 13.

From (3) we see that $2^{a_1+1}-1$ must be divisible by p_4. Setting $2^{a_1+1}-1 = kp_4$ we have

$$\begin{aligned} 9\cdot 2^{a_1-2}p_3 &= k(p_3+1)(p_4+1) \\ &= (p_3+1)(2^{a_1+1}-1+k) \end{aligned} \tag{4}$$

which may be written in the following form

$$2^{a_1-2}(p_3-8) = (k-1)(p_3+1). \tag{5}$$

Hence we must have $k > 1$ and $p_3 > 8$, and the only possible values of p_3 are 11 and 13. It is easily seen that $p_3 = 13$ does not satisfy (5). For $p_3 = 11$ we get from (5)

$$3\cdot 2^{a_1-2} = (k-1)12$$

or

$$2^{a_1-4} = k-1.$$

and since $2^5\cdot 2^{a_1-4} = kp_4 + 1$, $k = \dfrac{33}{32-p_4}$. Hence $p_4 = 31$, $k = 33$, $a_1 = 9$. The corresponding number is $m = 2^9\cdot 3\cdot 11\cdot 31$ which by trial is found to be multiply perfect.

II. Suppose $p_2 = 5$. In this case we should have

$$3 = \frac{2^{a_1+1}-1}{2^{a_1}}\cdot\frac{6}{5}\cdot\frac{p_3+1}{p_3}\cdot\frac{p_4+1}{p_4} \tag{6}$$

which gives

$$\left(\frac{p_3+1}{p_3}\right)^2 > \frac{5\cdot 2^{a_1-1}}{2^{a_1+1}-1} > \frac{5}{4}$$

or

$$\frac{p_3+1}{p_3} > \frac{1}{2}\sqrt{5}.$$

Hence the only possible value of p_3 is 7. Setting, as in the first case, $2^{a_1+1} - 1 = kp_4$ we get from (6)

$$35 \cdot 2^{a_1-4} = k(p_4 + 1)$$

or

$$3 \cdot 2^{a_1-4} = k - 1,$$

and $k = \dfrac{35}{32 - 3p_4}$. Hence $p_4 < 11$ and since $p_4 > 7$ this is absurd. Therefore p_2 cannot be equal to 5, and the only multiply perfect number of multiplicity 3 of the form $m = p_1^{a_1} p_2 p_3 p_4$ is the number $2^9 \cdot 3 \cdot 11 \cdot 31$.

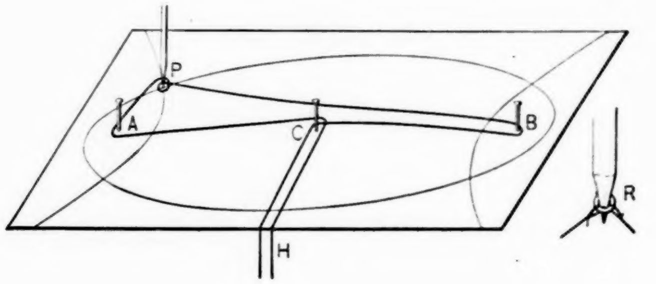
PURDUE UNIVERSITY, DECEMBER 1901.

A MECHANICAL CONSTRUCTION OF CONFOCAL CONICS.

By William R. Ransom.

By the following plan ellipses and hyperbolas whose foci are given may be drawn with very fair accuracy and very great readiness. The method was devised to facilitate plotting in elliptic coordinates, and it enables one to locate points in this system as freely as in the polar and rectangular systems.

Pins are driven at the foci, A, B, and a third at C, which is most conveniently taken midway between A and B. A string is passed loosely around the pencil point and the three pins in the way represented in the drawing. The pencil is then placed at the arbitrarily chosen point P, and the left hand, at H, draws the strings through the fingers until all slack is taken up. All is



then ready for drawing either the ellipse or the hyperbola through P. Holding the strings tightly at H, draw them down across the edge of the board, and the pencil traces a branch of the hyperbola down to the axis. Or hold the string tightly against the board between H and C, while the pencil slips laterally in its loop, and we get the ellipse.

The loop about the pencil may be dispensed with in drawing the ellipse, or, in drawing the hyperbola, may be replaced by a small circular link just large enough to slip the lead into, tied loosely, as shown at R, to the two ends of the string severed at P. With a little care, however, the tendency of the pencil to slide in the loop is not great enough to make it necessary to prevent this slipping by inserting such a link.

Tufts College, Massachusetts,
April, 1902.

ON SOPHUS LIE'S REPRESENTATION OF IMAGINARIES IN PLANE GEOMETRY.

By Percey F. Smith.

The first published paper of Sophus Lie appeared in the Transactions of the Academy of Christiania in February, 1869, and was concerned with the subject of this article.* So great a rôle does this memoir play in the scientific career of the author that a reference to it in the *Leipziger Berichte*, 1897, p. 726, elicits from him the observation that it formed the starting point of all his mathematical investigations.† In fact, the point of view there adopted leads in the most natural way to a point-line transformation of space which specialized gives the celebrated line-sphere transformation,‡ and generalized leads to a new duality defined by two *aequationes directrices* in two sets of point coordinates, upon which is based the monumental memoir in the fifth volume of the *Mathematische Annalen* (1872): Ueber Complexe, insbesondere Linien- und Kugel-Complexe, etc.

I propose to give a presentation of some of the results in the paper referred to above, and I am led to do this not only for the reason that the original lacks clearness and continuity, but also from the fact that Lie's work has been apparently overlooked by writers on this subject.§ Moreover, I think the method to be expounded possesses merits not shared by such other representations as I have seen. The essential difference consists in this, that Lie assumes the straight line as geometric element in the plane, while other representations are based upon point geometry. Thus the latter give a real line in space as image of a real or imaginary point in the plane,‖ while Lie's method also leads

* The full title is: Repräsentation der Imaginären der Plangeometrie. (Jeder plangeometrischer Satz ist ein besonderer Fall eines stereometrischen Doppel-Satzes in der Geometrie der Linien-Congruenzen). The paper appeared in two parts, pages 16–38, 107–146, the first part, with which we are especially concerned, having previously appeared in *Crelle*, vol. 70 (1869), p. 346. My references are to the Transactions.

† *Cf.* also Lie-Scheffers, *Geometrie der Berührungstransformationen*, vol. 1, p. 326, 448.

‡ First published in the note, Sur une transformation géométrique, *Comptes Rendus*, vol. 71 (1870), p. 579.

§ *Cf.* Coolidge, A purely geometric representation of all points in the projective plane, *Transactions Amer. Math. Soc.*, vol. 1 (1900), p. 182.

‖ For a very simple representation *cf.* the memoir by Duport, Sur un mode particulier de représentation des imaginaires, *Annales de l'école normale*, 1880, p. 301.

to a real line in space as representative of a real or imaginary *line* in the plane, each real element moreover being its *own* image. In Lie's representation there is then no change in *kind*.

To all such representations of the imaginaries in Plane Geometry there are of course two objections: *first*, that generalization to three dimensions is futile, and *secondly*, that the content of propositions in the plane is so greatly changed as to be of questionable value. The results of Lie's paper successfully meet the latter objection, I think; while perhaps all that can be said to justify a system such as von Staudt's, which substitutes a real substratum for the imaginaries in three dimensions, is summed up in Lindemann's opinion* that the *recognition of the possibility* of such a representation is a great advantage.† It is, however, right to add that von Staudt's work has led to a well developed theory of involutions.

As to the present presentation as compared with the original, I would say that the definition which I have adopted in §1 is purely projective, and the discussion proceeds in the simplest possible manner therefrom. From this the representation of Lie is deduced in §2 as a metrical special case. The latter, however, in his paper, makes use of an analytic definition only. He there gives, in fact, three different methods of representation, all defined analytically, and it is with the second of these (p. 32 *l. c.*) that we have to do. This is in substance identical with the first, as Lie points out, while the third method is merely defined and not developed. In the second part of the paper referred to, Lie develops synthetically the natural consequences of the representation.

We shall have to deal in our discussion with the four species of imaginary elements in space which present themselves in von Staudt's system, or also by allowing the equations of Analytic Geometry to contain complex coefficients. These are (1) *imaginary points* each lying on one real line, (2) *imaginary planes*, each containing one real line, (3) *imaginary lines of the first kind*, each lying in one real plane and containing one real point, (4) *imaginary lines of the second kind*, possessing neither a real plane nor a real point.

Since the representation introduces the right line as element of space, we shall have to do with some of the notions of line geometry.‡ Thus in §1

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* Clebsch-Lindemann: *Vorlesungen über Geometrie*, vol. 2, p. 130. An excellent presentation of von Staudt's theory is given in this volume, page 104.

† Compare also Professor Scott's critique: The status of imaginaries in pure geometry, *Bull. Amer. Math. Soc.*, ser. 2, vol. 7 (1900), p. 163.

‡ Koenigs, *La géométrie reglée.* Plücker, *Neue Geometrie des Raumes, etc.* Clebsch-Lindemann, *l. c.* p. 41.

we are led to the configuration of all lines of space intersecting two fixed lines, or, in the language of line geometry, to a linear line congruence of which the fixed lines are the directrices. The middle point of the common perpendicular of the directrices is the *centre*, and one-half of this shortest distance the *constant* of the congruence. I may refer to the text-books quoted for further details.

In von Staudt's system a linear line congruence is the real representative of an imaginary line of the second kind, the congruence consisting of all lines intersecting this line and the conjugate imaginary line.

1. Geometrical Definition of the Representation in Projective Form. Let σ be a real plane in space, the *fundamental plane*, and I an imaginary point, the *fundamental point*, not in σ. Also let L_0, the *fundamental line*, be the real line of the point I, and let S_0, the *fundamental pencil*, be the pencil of real lines in σ whose centre p_0 is the *foot* of the line L_0, that is the point in which L_0 meets σ.

Then any line l in σ may be represented by a real line L in space, viz., the real line of the plane containing I and l.

The line L is uniquely determined save when l belongs to the fundamental pencil, for in that case and in that case only is the plane of I and l real, containing as it does two real lines L_0 and l, and accordingly *L is now any line in a real plane containing the fundamental line.* If l is any other real line in σ, then L will coincide with l.

Conversely, every real line L in space is represented by a line l of σ, viz. the intersection of σ with the plane containing I and L.

The line l is uniquely determined save when L coincides with the fundamental line L_0, for then the plane of I and L becomes indeterminate and l is any line in σ through p_0.

We have therefore established in this way a (1, 1) correspondence between all the lines of σ and the real lines of space, the uniqueness failing only for the lines of the fundamental pencil in σ, and for the fundamental line in space.

It should be noted that l and L intersect in the real point on l.

Consider now in σ the locus of the first class, a point p. The ∞^2 lines of σ through p are represented by the real lines of ∞^2 planes passing through the line Ip. Each of these ∞^2 real lines, however, intersects not only Ip but also its conjugate line $J\bar{p}$, where J and $\bar{p}$ are the conjugate points of I and p respectively. Hence:

Any point p in σ is represented by a real linear line congruence whose directrices are the conjugate lines Ip and $J\bar{p}$.

The line of the congruence lying in σ is the real line through p. It should be remarked also that in general Ip and $J\bar{p}$ are imaginary lines of the second kind.

The congruence becomes special, *i. e.* the directrices intersect, when and only when the real line through p belongs to the fundamental pencil. For, since $\bar{p}$ also lies on this line, Ip and $J\bar{p}$ are now coplanar and accordingly intersect in a real point P. The congruence then consists of all lines through P and all lines in the plane of Ip and $J\bar{p}$, *i. e.* the plane containing P and the fundamental line L_0. Therefore:

A point in σ whose real line belongs to the fundamental pencil is represented by a special congruence consisting of all lines passing through a real point and all lines in the plane determined by that point and the fundamental line.

If p is real, then P coincides with it.

This association at the ∞^3 points of σ, whose real lines belong to the fundamental pencil, and the ∞^3 real points of space is very striking and of great importance in the sequel. The point p_0 in σ is represented by all lines intersecting the fundamental line L_0, *i. e.* by a special linear line complex whose axis is the fundamental line. This complex may be called the *fundamental complex*. The fundamental line L_0 belongs to every congruence representing a point p of σ, for it corresponds to the line $p\,p_0$.

Consider, now, a pencil of ∞^1 lines in σ through a point p, determined, for example, by a real parameter. Then we have the following:

THEOREM I. *The ∞^1 lines of a pencil in σ whose centre is p are represented in space by the generators of one system of a ruled quadric passing through I and J.*

Proof. The real line in space which represents a line of the given pencil in σ is the intersection of conjugate planes passing respectively through Ip and the conjugate line $J\bar{p}$.

Since the first plane intersects σ in a line of the given pencil, the ∞^1 given lines in σ evidently give rise to two projective pencils of planes through Ip and $J\bar{p}$, respectively, and, by a well known theorem, the line of intersection of corresponding planes of two projective pencils generates a ruled quadric. Finally, Ip and $J\bar{p}$ evidently belong to the generators of the other system, and therefore the quadric passes through I and J.

The quadric degenerates into two flat pencils when the real line through p belongs to the fundamental pencil and to the pencil considered. For, as explained above, in this case the congruence representing p becomes special.

Consider, next, a curve in σ of the second class, a conic C_2. This gives rise in space to a line congruence* of the second order; for through any point P in space will pass two lines of the congruence representing the tangents in σ to C_2 drawn through the foot of IP.

The cone with vertex I passing through C_2 is one of the focal surfaces of this congruence; for if consecutive lines of the latter intersect, then the corresponding consecutive tangents to C_2 do likewise, and this intersection is a point p on C_2; therefore the two lines of the congruence belong to a linear congruence and can intersect only on Ip. The entire focal surface consists of the above cone and the corresponding conjugate cone with vertex at J. The line congruence is then of the second order and fourth class. The real intersection of the focal surfaces, in general a skew curve of order four, corresponds to the points common to C_2 and the fundamental pencil. The tangents of this curve belong to the congruence.

If the conic C_2 passes through the foot p_0 of the fundamental line, then the real intersection of the focal surfaces degenerates into L_0 and a skew curve of the third order passing through I and J. Finally, if the tangents to C_2 through p_0 are real, then it is readily seen that the above cones have double contact on the line which represents the chord of contact of these tangents, and therefore their intersection consists of two conics.

Without prolonging the discussion let it suffice to state that a curve of class n in σ becomes in space a line congruence of order n whose focal surfaces are conjugate cones with vertices I and J. The tangents of the real intersection of these cones belong to the congruence.

2. Lie's Analytical Definition as a Metrical Special Case. Taking XZ as fundamental plane σ, and one of the circular points in the XY plane as I we obtain (using rectangular coordinates) Lie's definition. In fact, if

$$cx = bz + a \qquad (c \text{ real})$$

is any line l in XZ, then the plane through I and l is

$$c(x + iy) = bz + a,$$

and if

$$b = m + in, \qquad a = p + iq,$$

* In this connection the reader is referred to Picard, *Traité d'analyse*, vol. 1, p. 301.

the real line L of this plane is

$$cx = mz + p, \qquad cy = nz + q.$$

Conversely, given the equations of any real line L of space in this form, we obtain the equation of the corresponding line l in XZ by multiplying the second of these equations by i, adding, and then placing $y = 0$. The plane

$$c(x + iy) = bz + a$$

is real when and only when $c = o$ and b/a is real. Thus we see that the fundamental pencil consists of all real lines in XZ parallel to XX'. The line at infinity in XY is, of course, the fundamental line, as is evidenced by the obvious fact that for $c = o$ the equations of L reduce to those of the line in question, while l becomes $z =$ any constant. J is the other circular point in XY, and p_0 the point at infinity on XX'.

Any line in σ passing through a given point $p\ (\alpha, \beta)$ is given by

$$c(x - \alpha) = b(z - \beta),$$

where c and b are arbitrary constants, c being taken real. The plane containing this line and I is then :

(1) $$c(x + iy - \alpha) = b(z - \beta).$$

This equation defines ∞^2 planes (b complex) through the line Ip

(2) $$x + iy = \alpha, \quad z = \beta.$$

Let $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$. Now if $\beta_2 = 0$, *i. e.* if the real line through p belongs to the fundamental pencil, then the line (2) contains the real point $P(\alpha_1, \alpha_2, \beta_1)$ and is an imaginary line of the first kind. The real lines of the planes (1), if $c \neq 0$, pass through P: but for $c = 0$ the plane (1) becomes real, and we get all lines of the real horizontal plane through P, $z = \beta_1$.

If however $\beta_2 \neq 0$, (2) is always an imaginary line of the second kind, and the real lines of the ∞^2 planes (1) intersect (2) and also the conjugate line

(3) $$x - iy = \alpha_1 - i\alpha_2, \qquad z = \beta_1 - i\beta_2.$$

The lines (2) and (3) are accordingly the directrices of the linear congruence formed by all real lines in the planes (1). The shortest distance between them is $2i\beta_2$, and the middle point of this line is $(\alpha_1, \alpha_2, \beta_1)$, so that we have, adopting the nomenclature of Plücker,

THEOREM 2. *In Lie's analytic representation** *the imaginary point* $(a_1 + ia_2, \beta_1 + i\beta_2)$ *goes over into the* ∞^2 *lines of a linear line congruence whose centre is* (a_1, a_2, β_1) *and constant* $i\beta_2$.

The first method given by Lie in the paper referred to represents the imaginary point $(a_1 + ia_2, \beta_1 + i\beta_2)$ by the real point of space (a_1, a_2, β_1), to which is assigned a *weight* β_2. He very soon points out (page 20) that β_2 measures the constant of a line congruence, but adheres by preference to this representation throughout the paper. The ∞^3 points of the fundamental plane for which $\beta_2 = 0$, *i. e.* the points whose real lines constitute the fundamental pencil, he calls *null points*, which name I shall adopt. The important point is this, that each null point is represented by a special congruence whose centre is a real point P and whose plane is the plane PL_0. We may say, indeed, that the image of the null point is P itself, and conversely the special congruence constructed as aforesaid with P leads back to the original null point. Ambiguity enters only for the point p_0 which gives any point in the fundamental line.

THEOREM 3. *Lie's representation establishes a* (1, 1) *involutory correspondence between the lines of the plane and the real lines of space in which the fundamental configurations are, in the plane a pencil of real lines, and in space a line through the vertex of this pencil. By this is also set up a* (1, 1) *involutory correspondence of the* ∞^3 *points on the lines of this pencil (the null points of the plane), and the real points of space, and the uniqueness fails only for the vertex of the pencil.*

In this theorem are set forth the peculiar merits of Lie's method to which I have already referred.

It is to be observed that the combination of null point p and line containing p is represented by a real point P and a real line through P. Also the range of null points on any line l is projective with the real range on the corresponding line in space.

3. Correlation in the Plane and the Corresponding Point-Line Transformation of Space. On page 35 of Lie's paper is introduced the question of what arises in space when the representation is carried out on a correlation in the plane. The answer is immediate, for since the ∞^3 nul-points of σ go over into ∞^3 lines, then we obviously have a transformation T of the ∞^3 points of space into an assemblage of ∞^3 lines, *i. e.* a line complex

* *Lie, l. c.* p. 33.

Γ. The general features of T we may deduce by considering a polar reciprocation in σ with respect to a conic C_2 (Lie, §27). Since if p in σ reciprocates into l, then any null point of l corresponds to a line through p, we may state one fundamental property of T thus:

THEOREM 4. *If a point P in space describes a line L of the complex Γ, the corresponding line will turn around the point corresponding to L, i. e. will generate a cone whose vertex is that point.*

We next determine the degree of this cone (which we shall call a *complex cone*), which will give us the *order* of the line complex. If P describes an arbitrary line L the corresponding ∞^1 lines of Γ will represent the ∞^1 lines of a pencil in σ, *i. e.* by Theorem 1, will be generators of one system of a ruled quadric through I and J. Hence

THEOREM 5. *The complex cones of Γ are quadric cones through I and J, and therefore Γ is a line complex of order 2.*

Consideration of the relation of C_2 to the fundamental pencil in σ will lead to the singular properties of Γ. In the general case, p_0 reciprocates into a line l_0 which corresponds to a line L_0' of Γ. Then, as P describes L_0', the corresponding complex cone must degenerate into two planes; for the vertex must correspond to the base point p_0, *i. e.*, is any point of IJ. We assume this plane-pair *real*,* say E_1 and E_2, intersecting σ in l_1 and l_2. Now the ∞^3 lines of Γ are grouped in ∞^1 linear congruences representing the range on l_0 into which the lines of the fundamental pencil reciprocate. Two points of this range are null points, viz. the intersections of l_0 with l_1 and l_2; hence two *special* congruences belong to Γ; their centres are the intersections of L_0' with E_1 and E_2, and their planes are E_1 and E_2. If A_1, A_2 are the points of intersection of L_0' with E_1, E_2, then A_1 transforms into any line in E_2 while any point in E_2 gives a line through A_1. And since an arbitrary line L in space intersects E_1 and E_2, as P describes L the ruled quadric generated by the corresponding complex line passes through A_1 and A_2. The complex cones then also contain these points. We then easily obtain the following definition of Γ:

Given the tetrahedron A_1A_2IJ and a complex line L, then Γ consists of the generators of the ∞^2 ruled quadrics circumscribed to the tetrahedron and containing L, which belong to the same system as L.

For if p is the pole of l, and l' any line through p, then the ∞^1 null points

* This assumption is made in order to use our representation.

on l' give the generators of such a quadric; and all lines of Γ are determined by taking for l' any line through p. Since the definition is symmetrical with respect to the vertices of the tetrahedron, we see that every line through each vertex and in each face of the tetrahedron belongs to Γ. The complex Γ is then known to be a *tetrahedral complex.** The complex lines intersect the faces of the tetrahedron $A_1 A_2 IJ$ in a constant cross ratio.

THEOREM 6. *The general polar reciprocation in the fundamental plane yields, under Lie's representation, a transformation of the ∞^3 points of space into the lines of a tetrahedral complex, the points I and J being vertices of the fundamental tetrahedron.*†

The complex Γ will degenerate and the transformation take on special properties if we assume a special relation of C_2 and the fundamental pencil. For example, let the point p_0 be on C_2. Then it is readily seen that the complex has two singular points I, J and two singular planes intersecting in IJ. These planes intersect σ in the tangent to C_2 at p_0 and in the conjugate line. Without discussing the corresponding transformation, consider next what arises if p_0 reciprocates into a line l_0 belonging to the fundamental pencil. Then Γ degenerates into two special linear complexes, whose axes are the line IJ and a line in the plane IJl_0. The corresponding transformation of space becomes very simple. This case, however, as well as others of particular interest, appear if we do not limit ourselves to a reciprocation.‡

Consider then a correlation in σ such that p_0 transforms into a line l_0 of the fundamental pencil. Then a second line l_0' (in general imaginary) through p_0 will transform back into p_0. In the first instance the fundamental pencil correlates into a range on l_0 not including p_0, and the corresponding complex in space consists of ∞^1 special linear congruences whose centres lie upon a non-degenerate conic through I and J in the plane of IJ and l_0. That is, *the complex Γ is now a special quadratic complex consisting of all secants of a conic through I and J.* On the contrary the range determined on l_0' dual with the fundamental pencil contains one null point p_0, and accordingly the complex arising in space degenerates into two linear complexes one of which is

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* The reader is referred to Lie-Scheffers, *l. c.* p. 311, for an excellent discussion of this complex.

† This transformation was first remarked by Reye in 1868. *Cf. Geometrie der Lage*, zweite Abteilung, p. 125.

‡ Lie discusses only the case of a reciprocation, and thus the line-sphere transformation did not appear until later, as already remarked.

the special complex whose axis is IJ, *i. e.* the fundamental complex. Ambiguity in space is best avoided by regarding the transformation as a duality between two spaces r and R, such that the points of each correspond to the lines of a line complex in the other, these complexes being in R all secants of a fixed conic, and in r a general linear complex and the fundamental complex. We note here the essential characteristics of the line-sphere transformation; for the range on an arbitrary line in r must give rise to a ruled quadric containing the fixed conic, and if the latter is the imaginary circle at infinity, the quadric becomes a sphere.*

If we require that l'_0 also shall belong to the fundamental pencil, then the range on l_0 does contain p_0, and we readily see that the complex in either R or r degenerates into two special complexes, one of which is the fundamental complex.

The consideration of other special cases would not be without interest. Those enumerated may suffice in this place, however, and are moreover particularly signalled by Lie in all references to this subject.†

4. Duality of Space Defined by Two Bilinear Equations. Writing the equation of a correlation in XZ in the form

$$(4)\quad (Ax + Bz + C)X + (A'x + B'z + C')Z + A''x + B''z + C'' = 0,$$

or also

$$(5)\quad (AX + A'Z + A'')x + (BX + B'Z + B'')z + CX + C'Z + C'' = 0,$$

and confining ourselves to *null points*, *i. e.* to real values of z and Z, then writing $x + iy$ and $X + iY$ for x and X respectively, and separating (4) into real and imaginary parts, we obtain

$$(6)\qquad (a + ia')(X + iY) + (c + ic')Z + b + ib' = 0, \text{ or}$$

$$(7)\qquad \begin{cases} aX - a'Y + cZ + b = 0 \\ a'X + aY + c'Z + b' = 0, \end{cases}$$

that is since a, a', b, b', c, c' are *linear* in x, y, z, *two bilinear equations*. Denoting space by r or R according as we represent a point by x, y, z, or X, Y, Z, we may state

* Any point of R not on the fixed conic corresponds in r to a line of the general linear complex. But a point on that conic gives all lines in a plane through the axis of the fundamental complex. This representation of the general linear complex upon point space was first noticed by Noether, *Göttinger Nachrichten*, 1869, p. 305.

† *Cf. e. g. Mathematische Annalen*, vol. 5 (1872), p. 165; *Leipziger Berichte*, 1897, p. 728.

THEOREM 7. *Lie's representation applied to a correlation in the plane leads to a duality in space defined by two bilinear equations.*

Plücker discussed the duality in space defined by one *aequatio directrix*, but it remained for Lie to extend this notion to two equations, and the discovery came about precisely in the manner outlined above.*

The discussion of (7) is very simple, and the properties of the transformation deduced above are very easily established. The line $a = 0$, $a' = 0$ is L_0, and its points of intersection with the ruled quadric

$$\frac{b}{c} = \frac{b'}{c'}$$

are A_1 and A_2, and each of the points gives in R a plane $Z = \text{const.}$ Furthermore, the point I in R corresponds to the *plane* $a - ia' = 0$, *i. e.* the plane $A_1 A_2 J$ in r, etc.

Turning now to special cases, suppose p_0 in r, *i. e.* the point at infinity on XX', gives a line through that point itself. For this it is necessary and sufficient that $A = 0$. The line $a = 0$, $a' = 0$, now becomes the line at infinity in XY, and the singular tetrahedron reduces to IJ and two planes parallel to XY.

Suppose, however, that p_0 in r gives a real line l_0 through it; then from (5), A'/A'' must be *real*. And as we may take, without loss of generality, the line at infinity in XZ for l_0, A' becomes zero and (4), (5), and (7) reduce to

$$(8)\qquad (Bz + C)X + (B'z + C')Z + A''x + B''z + C'' = 0$$

$$(9)\qquad A''x + (BX + B'Z + B'')z + CX + C'Z + C'' = 0,$$

$$(10)\qquad \begin{cases} (a_0 z + a_1)X - (a'_0 z + a'_1)Y + (c_0 z + c_1)Z + b = 0 \\ (a'_0 z + a'_1)X + (a_0 z + a_1)Y + (c'_0 z + c'_1)Z + b' = 0, \end{cases}$$

where b and b' are still linear in x, y, z.

Taking now any line in r, $x = mz + p$, $y = nz + p$, substituting for x and y in (10), and eliminating z, we obtain a quadric whose trace upon the plane at infinity is readily found to be given by

$$(11)\qquad \begin{aligned} (a_1 a'_0)(X^2 + Y^2) + (c_1 c'_0)Z^2 + [(a_0 c_1) + (a'_0 c'_1)]\,YZ + \\ [(a'_0 c_1) + (a_1 c'_0)]'XZ = 0, \end{aligned}$$

* *Cf. Mathematische Annalen*, vol. 5 (1872), p. 143-157.

in which the parentheses denote determinants. This conic obviously passes through the circular points I and J.

The constants in (8) may be specialized with no loss of generality so that

$$B = i, \quad C = -1, \quad B' = -i, \quad C' = -1, \quad A'' = -1, \quad B'' = C'' = 0.$$

In fact, this amounts to choosing $(0, 0, 0)$, $(0, 0, 1)$, and $(1, 0, 0)$ in R to represent in r the axis ZZ' and the two minimum lines in xz passing through $(-1, 0, 0)$. We thus obtain from (10) the equations

$$(12) \qquad \begin{cases} X + zY + Z + x = 0, \\ zX - Y - zZ - y = 0, \end{cases}$$

while the conic (11) becomes

$$(13) \qquad X^2 + Y^2 - Z^2 = 0.$$

To obtain then from (12) the line-sphere transformation it suffices to replace Z by iZ, and these become

$$(14) \qquad \begin{cases} X + iZ + x + zY = 0, \\ z(X + iZ) - y - Y = 0. \end{cases}$$

These equations agree essentially with those adopted by Lie* by merely interchanging Z and Y.

From the equations (9) we see that the point at infinity on XX' in R gives in r an imaginary line in XZ when B/C is not real. The discussion of the previous section shows then that the line complex in r degenerates into a general linear complex and the special complex whose axis is IJ. We may readily verify this from the equations (12), for solving these for x and y, we have

$$(15) \qquad \begin{cases} x = -Yz - (X + Z), \\ y = (X - Z)z - Y. \end{cases}$$

Comparing these with the equations $x = rz + \rho$, $y = sz + \sigma$ as written by Plücker, we find

$$r = -Y, \quad \rho = -(X + Z), \quad s = X - Z, \quad \sigma = -Y,$$

* A detailed study of the line-sphere transformation is given in Lie-Scheffers, *Berührungstransformationen*, chap. 10, p. 411.

and for the remaining line-coordinate,*

$$\eta \equiv r\sigma - \rho s = X^2 + Y^2 - Z^2.$$

Then any point (X, Y, Z) not at infinity on the cone $X^2 + Y^2 - Z^2 = 0$ corresponds to a unique line of the general linear complex

$$r - \sigma = 0,$$

but the points excluded lead to the special linear complex

$$\eta = 0.$$

Finally, if the ratio B/C in (8) is real, $a_1 a_0' - a_0 a_1' = 0$, and the conic defined by (11) and the plane at infinity degenerates into two lines one of which lies in the plane XY; *i. e.* the line complex in either R or r consists of two special complexes one of which is the fundamental complex. As before, any point in R not on the degenerate conic at infinity corresponds to a line of the special complex in r whose axis is different from IJ. An example of this case is afforded by the polar reciprocation

$$(16) \qquad zZ + X + x = 0,$$

which corresponds to the transformation of space

$$(17) \qquad \begin{cases} X + x + zZ = 0, \\ \quad Y + y = 0. \end{cases}$$

Any point not at infinity in XY or XZ corresponds to a line parallel to the plane XZ.

Lie remarks that the transformation (17) is identical with that made use of by Euler and Ampère in the theory of partial differential equations of the first order, viz.

$$X = p, \quad Y + y = 0, \quad Z + z + px = 0, \quad p = \frac{\partial z}{\partial x},$$

and in fact (17) come from these by elimination of p.

Summing up, then, we may state

THEOREM 8. *The transformation of space established by two bilinear equations in the variables (x, y, z) and (X, Y, Z) leads to a duality of the spaces r and R in which the points of either correspond to the lines of a line complex in the other. In the general case, these complexes are general tetra-*

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* *Cf.* Clebsch-Lindemann, *l. c.* p. 44.

hedral complexes. The special cases (1) when the complex in r degenerates into a general and a special linear complex and that in R into all secants of a conic, and (2) when the complexes in both spaces degenerate into linear complexes, present themselves naturally and are among the most important of the special cases.

Of course, by a projective transformation in either r or R, or in both, the equations (7) assume the general bilinear form. A discussion of all special cases would seem to be not without interest.

5. Point-Line Transformation in General. In 1871 Lie attacked the problem of the determination of all algebraic transformations of space such that all points go over into the lines of a line complex, and the lines of a line complex into the points of space. He was unable, at that time, to solve the problem completely, and contented himself with the remark that all* (1, 1) transformations are given by bilinear equations. The solution was finally given in his paper Liniengeometrie und Berührungstransformationen, *Leipziger Berichte*, 1897, p. 687, and the results are recapitulated on page 740. It turns out that the only cases in addition to those defined as above are:

(1) The complex in r is a general linear, and in R a special quadratic complex consisting of all the tangents of a general quadric.

(2) Both complexes are special and consist of the tangents of developable surfaces, and one of these (or both as above) may become a special *linear* complex.

The first case arises in the simplest possible manner by using a point transformation given by Darboux† by which the secants of a conic become the tangents of a general quadric. This is done by, taking the case discussed above of a general linear complex in r‡ and the special complex in R of all secants of a conic, and transforming R by Darboux's point transformation.

Examples of the second type are derived by setting up a (1, 1) correspondence between the generating planes of two developables, and then assuming a duality such that a line in one plane shall correspond to a point in the other. The duality may be established, for example, by taking the con-

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* With this exception, that the general case when both complexes are linear and special is not completely represented. *Cf.* Lie, *Mathematische Annalen*, vol. 5 (1872), p. 167, footnote.

† Darboux, *Leçons sur la théorie générale des surfaces*, vol. 3, p. 493.

‡ The fundamental complex may be omitted in the statement for it arises from the ∞^1 points of the fundamental conic in R.

jugate of the given line with respect to a quadric, and the point of intersection of this conjugate with the corresponding plane of the other developable. In this way is obviously established a correspondence of the points of space and the tangents to a developable such that point and tangent through it go over into a like combination.

It is to be remarked that *both* complexes are general in one case only, viz. when both are tetrahedral complexes.

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MARCH, 1902.

NOTE ON THE GROUP OF ISOMORPHISMS OF A GROUP OF ORDER p^m.

BY G. A. MILLER.

THE first part of the following note is devoted to a study of some of the properties of the holomorphisms of a group of order p^m, p being any prime, which correspond to operators whose order is a power of p in the group of isomorphisms. In the second part an abelian subgroup of the group of isomorphisms of any abelian group of order p^m is determined. It is proved that this abelian subgroup is one of a series of conjugate subgroups which have in common the invariant operators of the group of isomorphisms.

1. Let P represent any group of order p^m, p being any prime, and let $P_1, P_2, \ldots . P_m$ represent any series of subgroups of orders $p, p^2, \ldots . p^m$ respectively such that P_{a-1} is contained in P_a, $a = 2, 3 \ldots . m$. The main object of this note is to consider all the holomorphisms of P which can be obtained on condition that every operator of P_a which is not in P_{a-1} corresponds to itself multiplied on the left by some operator of P_{a-1}.* It will first be proved that all such holomorphisms of P correspond to operators of order p^λ in the group of isomorphisms (I) of P; and, conversely, that each of the holomorphisms of P which corresponds to an operator of order p^λ in I is of this form.

If t_1, t_2 are any two operators of I which correspond to two such holomorphisms, then must $t_1 t_2$ have the same property. That is, to the totality of the possible holomorphisms for any series of subgroups such as $P_1, P_2, \ldots . P_m$ there corresponds a subgroup (I_1) of I. Let t_3 be any operator of I_1. In the holomorphism which corresponds to t_3 some operator (s) of P corresponds to $s_1 s$, where s_1 is commutative† with t_3. From this it follows that $t_3^{-n} s t_3^n = s_1^n s$, and hence the order of t_3 must be a power of p. Since t_3 is any operator of I_1 it follows that the order of I_1 is a power of p. When P is abelian this result may also be obtained by means of the known formula‡

$$t^{-n} s_a t^n = s_{a+n} s_{a+n-1}^n \cdot \cdot \cdot \cdot \cdot s_{a+n-r}^{\frac{n(n-1)\ldots(n-r+1)}{r!}} \cdot \cdot \cdot s_{a+1}^n s_a$$

* *Cf.* Burnside, *Theory of Groups of Finite Order*, 1897, p. 249.

† In the substitution group of degree p^m, determined by the two groups P and I. *Cf.* Burnside, *l. c.*, *p.* 227.

‡ *Bulletin of the American Mathematical Society*, vol. 7 (1901), p. 351.

whenever

$$t^{-1}s_\beta t = s_{\beta+1}s_\beta, \qquad \beta = a, a+1, \ldots\ldots a+n-1;$$

for s_{a+n} is the identity if $n > m-1$, and n may be so chosen that each of the exponents

$$n, \ldots\ldots \frac{n(n-1)\cdots\cdots(n-r+1}{r!}, \ldots\ldots n$$

is divisible by any power of p.

Frobenius has proved that in a group (P') of order $p^{m'}$ the total number of subgroups P'' of order $p^{m''}$ $(m'' < m')$ is $\equiv 1 \bmod p$. If P' is an invariant subgroup of our main group P of order p^m, a group P'' is invariant either under P or under one of a set of subgroups P'' conjugate under P. As the total number of these conjugates must be a multiple of p, it follows that the number of subgroups of order $p^{m''}$ which are contained in P' and invariant under P is $\equiv 1 \bmod p$.

We consider now any subgroup $\bar{I}$ of I, the order of $\bar{I}$ being $p^{\bar{n}}$, and prove that it is a subgroup I_1 related in the way explained in the first paragraph to at least one series of subgroups $P_1, P_2, \cdots P_m = P$ of P, in which, furthermore, every subgroup P_a is invariant under P. The group I_1 connected with an arbitrary series of subgroups is such a group $\bar{I}$, and it is connected also with a series of subgroups of the particular character just specified.

The subgroup $\bar{I}$ of I leaves invariant $P = P_m$, and at least one of its invariant subgroups (P_{m-1}) of order p^{m-1}, since its order is a power of p and the number of such subgroups is $\equiv 1 \bmod p$. Similarly it leaves invariant at least one (P_{m-2}) of the subgroups of P_{m-1} which are of order p^{m-2} and invariant under P_m. And so on. Thus the group $\bar{I}$ does leave invariant each one of a series of subgroups $P_1, \ldots P_m$ of the kind specified. But further it leaves it invariant in the way specified in the first paragraph, as one readily proves: for the quotient group P_{a+1}/P_a is of order p and its group of isomorphisms is of order $p-1$, which is prime to the order of $\bar{I}$. Such subgroups as I_1 depend, in general, upon P_1 and also upon the manner of selecting $P_2, P_3, \ldots\ldots, P_{m-1}$ after P_1 has been chosen. In particular, when P is cyclic, these subgroups can be chosen in only one way, while they can be chosen in a number of ways depending upon m when P is abelian and of type $(1, 1, 1, \ldots)$. In the latter case the totality of the holomorphisms for one series of subgroups such as $p_1, p_2, \ldots\ldots p_m$ is evidently the same as that for any other series, so that I_1, which is of order $p^{\frac{m(m-1)}{2}}$, has just as many conjugates under I as

there are different ways of selecting such a series; viz. $(p^m - 1)(p^{m-1} - 1) \ldots\ldots (p-1) \div (p-1)^m$. Each of these conjugates is therefore invariant in a subgroup (I_2) of I, whose order is $p^{\frac{m(m-1)}{2}}(p-1)^m$. Moreover, the quotient group I_2/I_1 is the direct product of m cyclic groups.

In the last example it was observed that, if any of the subgroups $P_1, P_2, \ldots . P_{m-1}$ is replaced by a different one, the corresponding subgroups of I will be conjugate, but not identical with I_1. This is clearly always the case when a holomorphism of P may be obtained by multiplying an operator of P_a by an arbitrary operator of P_{a-1}, $a = 2, 3, \ldots\ldots m$. When the last condition is satisfied, let I_2 represent the largest subgroup of I in which I_1 is invariant. We proceed to prove that I_2/I_1 is always a subgroup of the direct product of cyclic groups of order $p-1$.

In all the holomorphisms which correspond to I_2, each of the subgroups $P_1, P_2, \ldots\ldots, P_m$ corresponds to itself, and conversely, if each of these subgroups corresponds to itself in any holomorphism of P, this holomorphism must correspond to some operator of I_2. In all these holomorphisms the operators of P_a/P_{a-1} ($a = 2, 3, \ldots m$) correspond to some power of themselves. Let t_4, t_5 be any two operators of I_2 and consider the holomorphism which corresponds to the commutator of $t_4^{-1} t_5^{-1} t_4 t_5$. Since all the operators of P_a/P_{a-1} must correspond to themselves in this holomorphism, it follows from the preceding paragraph that the order of $t_4^{-1} t_5^{-1} t_4 t_5$ is a power of p. Hence I_1, which is composed of all the operators of I whose orders are powers of p, must include all the commutators of I. As the quotient group with respect to any invariant subgroup which includes the commutator subgroup is abelian,* I_2/I_1 must be abelian.† Furthermore, since the groups P_a/P_{a-1} are of order p and correspond to themselves in all these holomorphisms, I_2/I_1, *must be included in the direct product of cyclic groups of order* $p-1$.

It may be of interest to observe that a change in the series of subgroups $P_1, P_2, \ldots\ldots P_{m-1}$ does not necessarily affect I_1. For instance, when P is the direct product of two cyclic groups (C_1, C_2) of orders p^{m-1}, p respectively ($m > 2$), its group of isomorphisms (I) is of order $p^m(p-1)^2$.‡ In this case, let C_1 equal P_{m-1}. This determines the series $P_1, P_2, \ldots P_{m-1}$ and the corresponding I_1 is clearly of order p^{m-1}. The subgroup I_1 includes all

* *Quarterly Journal of Mathematics*, vol 28, 1896, p. 267.

† Wendt, *Mathematische Annalen*, vol. 55, 1901, p. 480.

‡ *Cf. Transactions of the American Mathematical Society*, vol. 2, 1901, p. 260.

the operators of I which satisfy the following conditions: the orders are powers of p, and they transform each of the cyclic subgroups of order P^{m-1} in P into itself. When P_{m-1} is replaced by any other cyclic subgroup of the same order, the remaining subgroups of the series $P_1, P_2, \ldots . P_{m-1}$ will not be changed, and the corresponding subgroup of I clearly satisfies the same condition as before, and hence it is identical with I_1.

2. In what follows it will be assumed that P is abelian. If p^{a_1} is the highest order of an operator in P, then it is possible to obtain $p^{a_1-1}(p-1)$ distinct holomorphisms of P by raising each one of its operators to the same power. It is known that these holomorphisms correspond to the $p^{a_1-1}(p-1)$ invariant operators of I.* We proceed to consider an important abelian subgroup of I which includes the characteristic subgroup composed of these invariant operators.

Let $H_1, H_2, \ldots . . H_n$ be any set of independent generating cyclic subgroups of P whose orders are $p^{h_1}, p^{h_2}, \ldots . p^{h_n}$ respectively; and consider any holomorphism of P in which each of these subgroups corresponds to itself. It is clearly possible to establish an arbitrary holomorphism of one of these subgroups with itself without affecting the holomorphism of any one of the other subgroups. Hence it follows that the totality of the holomorphisms of P in which each of these subgroups corresponds to itself must correspond in I to the direct product (A) of n cyclic groups of orders

$$p^{h_1-1}(p-1), \quad p^{h_2-1}(p-1), \quad \ldots . . \quad p^{h_m-1}(p-1)$$

respectively, whenever $p > 2$. When $p = 2$, the subgroup A is the direct product of a group of order 2^n and of type $(1, 1, 1) \ldots . .$ and n cyclic groups of orders $2^{h_1-2}, 2^{h_2-2}, \ldots . . 2^{h_m-2}$ respectively. The only case in which A reduces to the identity is when P is of type $(1, 1, 1 \ldots . .)$ and $p = 2$.

Let $S_1, S_2, \ldots . . S_n$ represent a set of generators of the cyclic subgroups $H_1, H_2, \ldots . . H_n$ respectively, and let $H'_1, H'_2, \ldots . H'_n$ represent a second set of independent generating cyclic subgroups of P. At least one of the latter subgroups (H'_a) is not generated by a single one of the operators of $S_1, S_2, \ldots . . S_n$. A generator of H'_a is therefore of the form $S_a^{a_1} S_\beta^{\beta_1} \ldots . . . .$, where at least two of the exponents $a_1, \beta_1, \ldots .$ differ from zero. As the subgroup A $(p > 2)$ includes some operators which transforms H'_a into itself, multiplied by some operator which is not found in H'_a, it

* *Cf.* the last foot-note.

follows that A *transforms into itself each member of only one of the possible sets of independent generating cyclic subgroups of P, whenever $p > 2$.*

From the preceding paragraph it follows that A has as many conjugates under I as there are different combinations of generating subgroups of P, whenever p is odd. In this case I contains no operators that transform A into itself besides those of A and those which transform the totality of the subgroups $H_1, H_2, \ldots\ldots H_n$ into itself, but permute some of them. The latter operators exist only when at least two of the independent generators of P are of the same order. Moreover, P contains no operator besides the identity which is invariant under A.

When $p = 2$, all the operators of order 2 in P are invariant under A, and hence A reduces to the identity when P is of type $(1, 1, 1, \ldots)$, as was observed above from another point of view. Since all the operators of A do not transform into itself any operator of P whose order exceeds 2, they cannot transform each of the subgroups $H'_1, H'_2, \ldots\ldots H'_n$ into itself unless the order of no more than one factor in the product $S_\alpha^{\alpha_1} S_\beta^{\beta_1} \ldots\ldots$ exceeds 2 for every H'_α. This condition is clearly sufficient as well as necessary.

All the conjugates of A have the $p^{\alpha_1 - 1}(p-1)$ invariant operators of I in common for all values of p, since each of these operators transforms every subgroup of P into itself.

Moreover, it is easy to prove that every operator (t) which is found in all the conjugates of A is also included among these invariant operators of I. From the fact that the product $S_1 S_2 \ldots S_n$ may be used as an independent generator of P it follows that

$$t^{-1} S_1 S_2 \ldots\ldots S_n t = (S_1 S_2 \ldots S_n)^\beta \quad \text{and} \quad t^{-1} S_i t = S_i^{\beta_i} \quad i = 1, 2, \ldots, n.$$

Hence

$$(S_1 S_2 \ldots S_n)^\beta = S_1^{\beta_1} S_2^{\beta_2} \ldots S_n^{\beta_n}.$$

We may therefore set $\beta_1 = \beta_2 = \cdots = \beta_n\ \beta$. Since t transforms each generator of P into its βth power, it also transforms each operator of P into this power: that is, *the $p^{\alpha_1 - 1}(p-1)$ invariant operators of I are the only ones which are common to all the conjugates of A under I.*

STANFORD UNIVERSITY,
JANUARY, 1902.

EVALUATION OF SLOWLY CONVERGENT SERIES.*

By L. D. Ames.

1. General Theory. Any convergent series

$$S = \tfrac{1}{2}u_1 + \tfrac{1}{2}u_2 + u_3 + \cdots \tag{1}$$

can be written in the form:

$$S = \tfrac{1}{2}u_1 + \tfrac{1}{2}[(u_1 + u_2) + (u_2 + u_3) + \cdots]. \tag{2}$$

Repeating the process on the series in brackets we have:

$$S = \tfrac{1}{2}u_1 + \tfrac{1}{4}(u_1 + u_2) + \tfrac{1}{4}[(u_1 + 2u_2 + u_3) + (u_2 + 2u_3 + u_4) + \cdots], \tag{3}$$

$$S = \tfrac{1}{2}u_1 + \tfrac{1}{4}(u_1 + u_2) + \tfrac{1}{8}(u_1 + 2u_2 + u_3)$$
$$+ \tfrac{1}{8}[(u_1 + 3u_2 + 3u_3 + u_4) + (u_2 + 3u_3 + 3u_4 + u_5) + \cdots], \tag{4}$$

and in general:

$$S = \frac{1}{2}u_1 + \frac{1}{2^2}(u_1 + u_2) + \cdots\cdots$$
$$+ \frac{1}{2^k}\left[u_1 + (k-1)u_2 + \frac{(k-1)(k-2)}{1\cdot 2}u_3 + \cdots + u_k\right] + R_k, \tag{5}$$

where

$$R_k = \frac{1}{2^k}\Big[(u_1 + ku_2 + \frac{k(k-1)}{1\cdot 2}u_3 + \cdots\cdots + u_{k+1})$$
$$+ (u_2 + ku_3 + \frac{k(k-1)}{1\cdot 2}u_4 + \cdots + u_{k+2}) + \cdots\Big]. \tag{6}$$

It will be proved in §3 that

$$\lim_{k=\infty} R_k = 0.$$

Assuming this for the present we see that (1) may be written in the form:

$$S = \frac{1}{2}u_1 + \frac{1}{2^2}(u_1 + u_2) + \frac{1}{2^3}(u_1 + 2u_2 + u_3) + \cdots\cdots. \tag{7}$$

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* Read before the American Mathematical Society at its meeting, April 26, 1902.

It turns out, in a large variety of slowly convergent series in which the positive and negative terms are about evenly balanced in number and aggregate value throughout the series, that series (7) converges rapidly and the error made by stopping at any term is less than the last term added. In practical computation this fact will often be sufficiently evident from a simple inspection of the series. The question of finding an upper limit to the error will be considered later. See §5.

2. Illustrative Examples. We will now illustrate the method described in the last section by applying it to some numerical examples, and in doing so we will indicate a number of slight modifications which are often advantageous.

Example I. $S(x) = \sin x - \dfrac{\sin 2x}{2} + \dfrac{\sin 3x}{3} - \ldots .$

To evaluate this series when $x = 50°$ we may tabulate the work as follows:

n	u_n	$u_n + u_{n+1}$	$u_n + 2u_{n+1} + \ldots$	$u_n + 3u_{n+1} + \ldots$	$u_n + 4u_{n+1} + \ldots$	$u_n + 5u_{n+1} + \ldots$	$u_n + 6u_{n+1}$
1	**+ .7660**	**+ .2736**	**— .0521**	**— .1256**	**— .0493**	**+ .0308**	**+ .0446**
2	— .4924	— .3257	— .0735	+ .0763	+ .0801	+ .0138	
3	+ .1667	+ .2522	+ .1498	+ .0038	— .0663		
4	+ .0855	— .1024	— .1460	— .0701			
5	— .1878	— .0436	+ .0759				
6	+ .1443	+ .1195					
7	— .0248						

The first terms of the successive columns correspond to the terms of (7).

$S = \frac{1}{2} \times .7660 + \frac{1}{4} \times .2736 - \frac{1}{8} \times .0521 + \cdots + \frac{1}{128} \times .0446 + \cdots = .4364 +.$

But $S(x) = \dfrac{x}{2}$, therefore the correct result is .43633. For $x = 10°$ the same series gives a result correct to four places by the use of four terms.

Example II. $S = \dfrac{1}{\log 2} - \dfrac{1}{\log 3} + \dfrac{1}{\log 4} - \ldots .$

We compute the table as before:

n	u_n	$u_n + u_{n+1}$	$u_n +$ $2u_{n+1} + \cdot\cdot$	$u_n +$ $3u_{n+1} + \cdot\cdot$	$u_n +$ $4u_{n+1} + \cdot\cdot$	$u_n +$ $5u_{n+1} + \cdot\cdot$
1	**1.4427**	.5325	.3436	.2547	.2026	.1683
2	**— .9102**	— .1889	— .0889	— .0521	— .0343	
3	**.7213**	**.1000**	**.0368**	**.0178**		
4	— .6216	— .0632	— .0190			
5	.5581	.0442				
6	— .5139					

A slight inspection shows that it is best to sum two terms separately, using the terms in heavy type. The terms above those used need not have been computed.

$$S = 1.4427 - .9102 + \left\{\tfrac{1}{2} \times .7213 + \tfrac{1}{4} \times .1000 + \cdot\cdot\right\} = .9239 + .$$

This result is correct to the third place of decimals, as can be proved by §5.

Example III. The reader may compute

$$S(x) = P_0(\mu) - \tfrac{1}{2}\, P_2(\mu)x + \tfrac{1\cdot 3}{2\cdot 4}\, P_4(\mu)x^2 - \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\, P_6(\mu)x^3 + \cdot\cdot\cdot,$$

for x equal to or near 1, and μ any desired value. (*Cf.* Byerly, *Fourier's Series and Spherical Harmonics*, p. 152.)

Example IV. $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdot\cdot\cdot\cdot.$

Since the more slowly the original series converges the more rapidly the resulting series converges, we sum ten terms separately and apply the transformation (7) to the next eleven. We obtain $\dfrac{\pi}{4} = .785398163$, correct to nine places (*cf.* §5).

Example V. $S = 1 - \dfrac{1}{2^p} + \dfrac{1}{3^p} - \dfrac{1}{4^p} + \cdot\cdot\cdot,\quad p = 1.001.$

Sum four terms separately, and apply the transformation to the next seven.

$$S = .693309 + R, \qquad R < .000003. \qquad (§5.)$$

Example VI. $S = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots, \quad p = 1.001.$

$$S = \frac{2^p}{2^p - 1}\left(1 - \frac{1}{2^p} + \frac{1}{3^p} \quad \cdots\right) = 1000.563 + R, \qquad \text{(Ex. V.)}$$

$$R < .004.$$

Example VII. $S = 1 + \frac{5 \cdot 7}{8 \cdot 10} + \frac{5 \cdot 7 \cdot 9 \cdot 11}{8 \cdot 10 \cdot 12 \cdot 14} + \cdots.$

$$S = 6\left[1 - \frac{5}{6} + \frac{5 \cdot 7}{6 \cdot 8} - \frac{5 \cdot 7 \cdot 9}{6 \cdot 8 \cdot 10} + \cdots\right] = 3.341 + R, \qquad R < .006$$

by the use of four terms (*cf.* §5).

Example VIII. $S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} \cdots.$

Group the terms thus:

$$S = \left(1 + \frac{1}{3}\right) - \frac{1}{2} + \left(\frac{1}{5} + \frac{1}{7}\right) - \frac{1}{4} + \cdots.$$

Computing the table as in Ex. I, the fifth column is seen to consist of positive terms. It will not be useful to obtain another column. By (5):

$$S = \tfrac{1}{2} \times 1.333 + \tfrac{1}{4} \times .833 + \tfrac{1}{8} \times .676 + \tfrac{1}{16} \times .612 + \tfrac{1}{16}[.593 + .013 + .030$$

$$+ .003 + \cdot \cdot] = 1.037 + \cdot$$

Here the degree of accuracy is not evident. There is strong probability that the result is much closer to the true value than could have been reached by simply adding terms. This example illustrates the limitations of the method: also how it will usually become evident if the method cannot be profitably employed in its entirety, and how it may still be more or less useful in a modified form.

3. Proof of Convergence. We will now complete the proof of (7) by showing that the remainder R_k of (6) approaches zero as k becomes infinite.

$$R_k = R'_k + R''_k + R'''_k.$$

where

$$R'_k = \frac{1}{2^k}\Bigg[u_1 + (1+k)u_2 + \left(1 + k + \frac{k(k-1)}{1\cdot 2}\right)u_3 + \cdots$$
$$+ \left(1 + k + \frac{k(k-1)}{1\cdot 2} + \cdots + \frac{k(k-1)\cdots(k-r+2)}{1\cdot 2\ldots(r-1)}\right)u_r\Bigg].$$
$$R''_k = \frac{1}{2^k}\Bigg[\left(1 + k + \frac{k(k-1)}{1\cdot 2} + \cdots + \frac{k(k-1)\cdots(k-r+1)}{1\cdot 2\cdots r}\right)u_{r+1} + \cdots$$
$$+ \left(1 + k + \frac{k(k-1)}{1\cdot 2} + \cdots + \frac{k(k-1)}{1\cdot 2} + k\right)u_k\Bigg],$$
$$R'''_k = u_{k+1} + u_{k+2} + \cdots\cdots.$$

Our theorem will be proved if we can show that a positive ϵ having been chosen at pleasure, a positive m exists such that when $k > m$:

$$|R'_k| < \frac{\epsilon}{3}, \qquad |R''_k| < \frac{\epsilon}{3}, \qquad |R'''_k| < \frac{\epsilon}{3}.$$

Since the u-series converges, m''' can be chosen so that $|R'''_k| < \epsilon/3$ when $k > m'''$.

Similarly choose m'' so that when $k > r > m''$:

$$|u_{r+1} + u_{r+2} + \cdots\cdots u_k| < \epsilon/3.$$

R''_k is of the form:

$$\eta_{r+1}u_{r+1} + \eta_{r+2}u_{r+2} + \cdots\cdots + \eta_k u_k,$$

where

$$0 < \eta_{r+1} < \eta_{r+2} < \cdots\cdots < \eta_k < 1.$$

Hence, by a well-known lemma of Abel's,* when $k > r > m''$,

$$|R''_k| < \epsilon/3.$$

Let M be a positive constant greater than the absolute value of any term of the u-series. Then

$$|R'_k| < \frac{Mr}{2^k}\left[1 + k + \frac{k(k-1)}{1\cdot 2} + \cdots + \frac{k(k-1)\cdots(k-r+2)}{1\cdot 2\cdots(r-1)}\right]$$
$$< \frac{Mr^2}{2^k}\left[\frac{k(k-1)\cdots\cdots(k-r+2)}{1\cdot 2\cdots\cdots(r-1)}\right], \qquad \text{if } k > 2r,$$
$$< \frac{Mr^2}{2^k}\,k^{r-1}.$$

* *Cf.* for instance Tannéry: *Théorie des fonctions d'une variable*, p. 95.

Put $k = r^2$, then $$|R'_k| < \frac{M r^{2r}}{2^{r^2}} = M\left(\frac{r^2}{2^r}\right)^r.$$

But $$\lim_{r=\infty} \frac{r^2}{2^r} = 0. \qquad \therefore \lim_{r=\infty}\left(\frac{r^2}{2^r}\right)^r = 0. \qquad \therefore \lim_{k=\infty} R'_k = 0.$$

Therefore m' can be chosen so that $|R'_k| < \epsilon/3$ when $k > m'$.

By taking m greater than the largest of the three quantities m', m'', m''', the truth of the theorem follows at once.

4. A Shorter Method of Computation. By subtracting R_k, in the form given in §3, from S we obtain:

$$S = \frac{1}{2^k}\left[\lambda_1 u_1 + \lambda_2 u_2 + \cdots \lambda_k u_k\right] + R_k, \tag{8}$$

where

$$\lambda_r = 1 + k + \frac{k(k-1)}{1\cdot 2} + \cdots \frac{k(k-1)\cdots(r+1)}{1\cdot 2\cdots(k-r)}. \tag{9}$$

The λ's are the same for all series and may be computed once for all. A few sets are tabulated:

k	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}
4	15	11	5	1							
7	128	120	99	64	29	8	1				
11	2047	2036	1981	1816	1486	1024	562	232	67	12	1

By the use of this table Ex. II, §2 would be computed as follows:

$$S = 1.4427 - .9102 + \frac{1}{2^4}(15\times.7213 - 11\times.6216 + 5\times.5581 - .5139) + R$$
$$= .9237 + R.$$

Unless we know from other considerations (§5) an upper limit to R the method used in §2 is usually more convenient, since in that method we can observe the rate of convergence.

5. Determination of an Upper Limit for the Error. Let us confine ourselves henceforth to the alternating series:

$$S = v_1 - v_2 + v_3 - v_4 + \cdots, \qquad v_n > 0. \tag{10}$$

Form the successive orders of differences of the absolute value series $v_1 + v_2 + v_3 + \cdots$. If we denote the general term by $\Delta^r v_n$, the series (7) becomes:

$$S = \frac{1}{2} v_1 - \frac{1}{2^2}\Delta^1 v_1 + \frac{1}{2^3}\Delta^2 v_1 - \cdots\cdots + (-1)^{r-1}\frac{1}{2^r}\Delta^{r-1} v_1 + \cdots. \tag{11}$$

In some cases the differences can be computed in general terms. Thus let:

$$S = 1 - \frac{a}{b} + \frac{a(a+1)}{b(b+1)} - \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \cdots\cdots, \qquad a < b. \tag{12}$$

Computing the differences and substituting in (11):

$$S = \frac{1}{2} + \frac{1}{2^2}\frac{b-a}{b} + \frac{1}{2^3}\frac{(b-a)(b-a+1)}{b(b+1)} + \cdots. \tag{13}$$

The ratio of any term to the preceding term is less than $\frac{1}{2}$. Hence the error made by stopping at any term is less than the last term added. The series in §2, Exs. IV and VII are of this form.

Theorem. *If there is a function $f(x)$ which has continuous derivatives of the first $k+1$ orders when $x \geqq 1$; and if the derivatives of even order are positive, those of odd order negative; and $f(n) = v_n = (-1)^{n+1}u_n > 0$; then*

$$|R_k| < \frac{1}{2^k}|u_1|.$$

Proof. Form the sequence

$$f(x_0), \quad f(x_0 + \Delta x), \quad f(x_0 + 2\Delta x), \cdots.$$

It can be proved* that the rth difference quotient $\Delta^r f(x)/\Delta x$ converges uniformly to the value $d^r f(x)/dx^r$ as Δx approaches zero. Hence a positive number δ may be chosen so that for all values of $x \geqq 1$, when $\Delta x < \delta$, $\Delta^r f(x)/\Delta x^r$ and $d^r f(x)/dx^r$ have the same sign. Now choose $\Delta x < \delta$ so that Δx is an aliquot part of unity, *i. e.*, $m\Delta x = 1$. Then the v-series may be formed by taking every mth term of the above sequence, if x_0 is properly chosen. It is easily shown that any difference of the v-series is the sum of positive multiples of dif-

* *Cf.* for instance, Harnack, *Die Elemente der Differential- und Integralrechnung*, §32, where the proof can easily be made to meet the question as to the uniformity of the convergence.

ferences of the same order of this series. Hence $\Delta^r v_n$ is positive when r is even and negative when r is odd. It follows that

$$|\Delta^k v_n| > |\Delta^k v_{n+1}|.$$

But, from (6),

$$R_k = \frac{1}{2^k} (\Delta^k v_1 - \Delta^k v_2 + \Delta^k v_3 \cdot \cdot \cdot \cdot).$$

Hence

$$|R_k| < \frac{1}{2^k} |\Delta^k v_1| < \frac{1}{2^k} |u_1|.$$

By the use of this theorem we can prove that the results of Exs. II, IV, V, VI are correct.

6. Application to Divergent Series. If our method be applied to certain divergent series it renders them convergent. This fact becomes of peculiar interest when we apply the method to power series, as we thus get an analytic continuation of the function originally represented. But similar results may be reached by the method of conformal transformation.*

As this treatment of this phase of the subject is simpler, we will not enter upon the discussion here.

Harvard University,
Cambridge, Massachusetts,
April, 1902.

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* Painlevé has given a general statement of the method; Paris Thesis, 1887; *Annales de la Faculté des Sciences de Toulouse*, vol. 2 (1888), p. B.1. A number of special developments are worked out in detail by E. Lindeiöf, Acta societatis scientiarum Fennicae, vol. 24 (1898), No. 7.

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ANNALS OF MATHEMATICS.

Published in October, January, April, and July, under the auspices of Harvard University, Cambridge, Mass., U. S. A.

Cambridge: Address *The Annals of Mathematics*, 2 University Hall, Cambridge, Mass., U. S. A. Subscription price, $2 a volume (four numbers) in advance. Single numbers, 75c. All drafts and money orders should be made payable to Harvard University.

London: Longmans, Green & Co., 39 Paternoster Row. Price, 2 shillings a number.

Leipzig: Otto Harrassowitz, Querstrasse 14. Price, 2 marks a number.

PRINTED BY THE SALEM PRESS CO., SALEM, MASS., U. S. A.